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Interpolating Surfaces: High Order Convergence Rates and Their Associated Designs, with Application to X-Ray Image Reconstruction

by

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Abstract

We consider the problem of interpolating a function on the unit square from values of the function given on a set (called the "design") of N discrete points. ~~We obtain~~ Then some new results are obtained on better ways to distribute these N points so as to minimize the maximum interpolation error. W. J. Gordon [8, Approximation With Special Emphasis on Spline Functions, I. Schoenberg, ed. 1969], showed that an improvement on the tensor product design $\{(x,y)\} = \{\frac{i}{n}, \frac{j}{n}, i,j = 1,2,\dots,n\}$, of $N = n^2$ interpolation points could be made by using a design that is the union of two tensor product designs: $\{(x,y)\} = \{\frac{i}{n}, \frac{j}{n^2}, i = 1,2,\dots,n, j = 1,2,\dots,n^2\} \cup \{\frac{i}{n^2}, \frac{j}{n}, i = 1,2,\dots,n^2, j = 1,2,\dots,n\}$. Here $N = 2n^3 - n^2$ since the intersection of these tensor product designs has n^2 points. Gordon showed that where interpolation on the original tensor product design has a convergence rate $O(N^{-p/2})$ for some p with respect to functions with specified smoothness properties, the new design will have an associated convergence rate of $O(N^{-2p/3})$. We construct the ℓth design which is the union of $\ell+1$ tensor product designs: $\{(x,y)\} = \bigcup_{k=1}^{\ell+1} \frac{i}{n^k}, \frac{j}{n^{\ell+2-k}}, i = 1,2,\dots,n^k, j = 1,2,\dots,n^{\ell+2-k}$. Here the ℓth design has $N = (\ell+1)n^{\ell+2} - \ell n^{\ell+1}$ distinct points, and we show that the associated convergence rate is $O(N^{-[(\ell+1)p/(\ell+2)]})$. Some exact theorems to this effect are proved for convergence of minimal norm interpolation in tensor product reproducing kernel Hilbert spaces. An application to the reduction of X-ray dosage in the reconstruction of human internal structure from X-ray projections is suggested. Applications to the design of certain (other) indirect sensing experiments are also noted.

1. Introduction

We consider the problem of interpolating a surface on the unit square from values of the surface given on a set (to be called the "design") of N discrete points. We obtain some new results on better ways to distribute these N points so as to minimize the interpolation error.

Let $f(x,y)$ be a function defined on $[0,1]\times[0,1]$ with the property that

$$\frac{\partial^{p+q}}{\partial x^p \partial y^q} f(x,y) \qquad \text{exists and is continuous} \tag{1.1}$$

for $0 \leq p \leq 2m$ and $0 \leq q \leq 2m$. Convergence properties of piecewise polynomial, including bivariate spline, interpolation are known for such functions when interpolation is carried out over a tensor product design. By a tensor product design is meant a set of $N = N_1 N_2$ points in the unit square of the form $\{(x_i, y_j)\ i = 1,2,\ldots,N_1,$ $j = 1,2,\ldots,N_2\}$ where $0 \leq x_1 < x_2 < \ldots < x_{N_1} \leq 1$, $0 \leq y_1 < \ldots < y_{N_2} \leq 1$. See deBoor and Fix [6] Lyche and Schumaker [14]. Specializing to $N_1 = N_2 = n$, $N = n^2$, in general one has, with appropriate interpolants $\hat{f}$,

$$\sup_{x,y} |f(x,y) - \hat{f}(x,y)| = O\left(\frac{1}{n^{2m}}\right) = O\left(\frac{1}{N^m}\right)$$

as $n \to \infty$. This is the same rate as is achieveable in univariate interpolation - if $g(z)$, $z \in [0,1]$ has 2m continuous derivatives, then, for example, local Lagrange (polynomial) interpolation of degree 2m-1 through 2m adjacent points $\frac{i}{n}, \frac{i+1}{n}, \ldots, \frac{i+2m-1}{n}$ gives a remainder term $O\left(\frac{1}{n^{2m}}\right)$, see [12, p.190], and the same rates are obtainable for various types of spline interpolation (see, e.g. [22]).

It will be convenient here to report our results for functions in a reproducing kernel Hilbert space (r.k.h.s.) of functions which satisfy (1.1) for $p,q \le 2m-1$ and the slightly weaker condition

$$\int_0^1 \int_0^1 \left(\frac{\partial^{4m}}{\partial x^{2m} \partial y^{2m}} f(x,y)\right)^2 dxdy < \infty. \tag{1.2}$$

Convergence results for certain types of bivariate spline interpolation on the tensor product design $T_{N,0} \equiv \{\frac{i}{n}, \frac{j}{n}\}_{i,j=1}^{n}$, $N = n^2$, for functions in an r.k.h.s., satisfying (1.2) have been obtained by Mansfield [17, Eq. (4.2)], who gives

$$\sup_{x,y \in [0,1]} |f(x,y) - \hat{f}(x,y)| = O\left(\frac{1}{n^{2m-1/2}}\right) = O\left(\frac{1}{N^{m-1/4}}\right) .$$

and it appears that this is the best possible rate under the given conditions, for the tensor product design $T_{N,0}$.

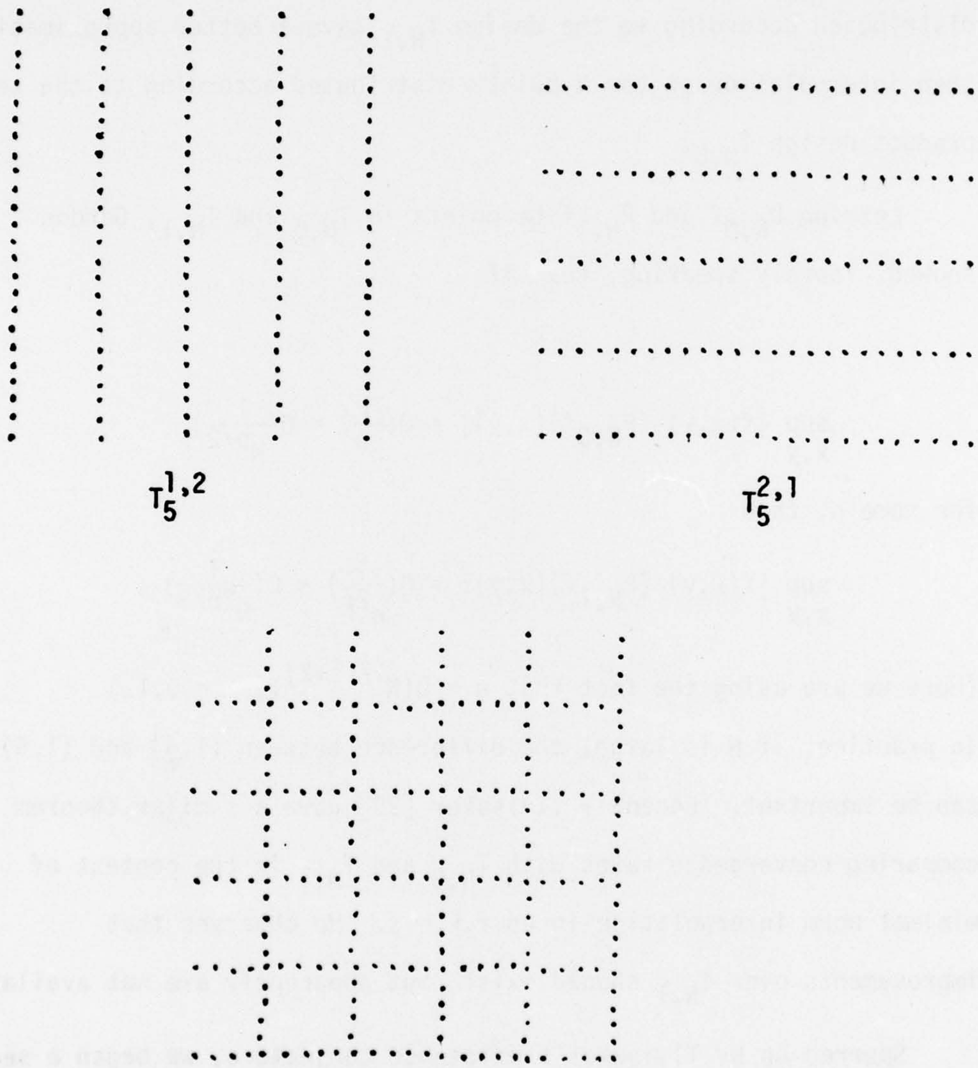
For many years we have been intrigued by W. J. Gordon's results [8] with the so-called blending function splines. Gordon used designs of the form

$$T_{N,1} = T_n^{1,2} \cup T_n^{2,1}$$

where

$$T_n^{j,k} = \left\{\frac{\xi}{n^j} \quad \frac{\eta}{n^k}\right\}_{\xi=1,\eta=1}^{n^j \quad n^k} \tag{1.3}$$

The two tensor product designs $T_n^{1,2}$ and $T_n^{2,1}$ each have n^3 points and their intersection, $T_n^{1,2} \cap T_n^{2,1}$, is $T_n^{1,1} = \{\frac{\xi}{n}, \frac{\eta}{n}\}_{\xi=1,\eta=1}^{n \quad n}$, which has n^2 points, so that their union $T_{N,1}$ has $N = 2n^3 - n^2$ distinct points. See Figure 1.1, which depicts the designs $T_n^{1,2}$, $T_n^{2,1}$ and $T_{N,1}$ for $n = 5$, (thus N=225). Figure 1.2, taken from Gordon, schematically describes the interpolation of f over part of $T_{N,1}$. Gordon showed



$T_5^{1,2}$ $T_5^{2,1}$ $T_{225,1}$

Figure 1.1

The designs $T_n^{1,2}$, $T_n^{2,1}$ and $T_{N,1}$, n=5, N=225

theoretically and by numerical example with bicubic spline interpolation, that interpolation to appropriately smooth functions on the N points distributed according to the design $T_{N,1}$ gave a better approximation than interpolation on the N points distributed according to the tensor product design $T_{N,0}$.

Letting $P_{N,0}f$ and $P_{N,1}f$ be points in $T_{N,0}$ and $T_{N,1}$, Gordon showed, loosely speaking, that if

$$\sup_{x,y} |f(x,y)-(P_{N,0}f)(x,y)| = 0(\frac{1}{n^p}) = 0(\frac{1}{N^{p/2}}) \tag{1.4}$$

for some p, then

$$\sup_{x,y} |f(x,y)-(P_{N,1}f)(x,y)| = 0(\frac{1}{n^{2p}}) = 0(\frac{1}{N^{2p/3}}) \ . \tag{1.5}$$

(Here we are using the fact that $n = 0(N^{1/(\ell+2)})$, $\ell = 0,1$.) In practice, if N is large, the difference between (1.4) and (1.5) can be important. Recently Ylvisaker [28] gave a similar theorem comparing convergence rates with $T_{N,0}$ and $T_{N,1}$ in the context of minimal norm interpolation in an r.k.h.s. He observes that improvements over $T_{N,1}$ should exist, but apparently are not available.

Spurred on by Ylvisaker's existence conjecture, we began a search for designs that improve upon the rate $0(\frac{1}{N^{2p/3}})$. This search was successful, and in this paper we provide a sequence of designs $T_{N,\ell}$, $\ell = 0,1,\ldots$ with the following property. When the tensor product design $T_{N,0}$ results in a convergence rate of $0(\frac{1}{n^p}) = 0(\frac{1}{N^{p/2}})$, the design $T_{N,\ell}$ with ℓ fixed, has $N = (\ell+1)n^{\ell+2}-\ell n^{\ell+1}$ points, and results in the convergence rate

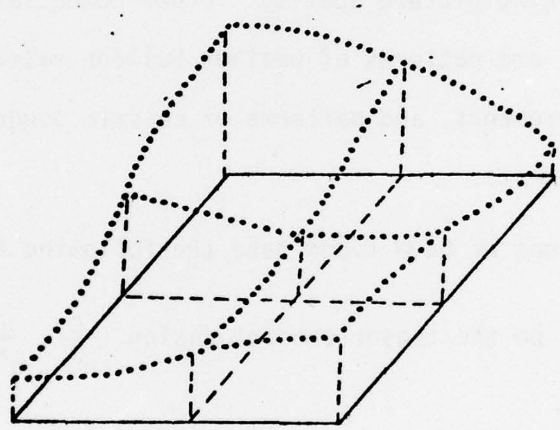


Figure 1.2

Interpolation over part of $T_{N,1}$

$$O\left(\frac{1}{n^{(\ell+1)p}}\right) = O\left(\frac{1}{N^{\frac{\ell+1}{\ell+2}p}}\right),$$

as $n \to \infty$. Considerable improvement in some practical problems appears to be possible as a result.

Later in this paper we suggest how this result might be used to reduce the X-ray dosage in computerized image reconstruction while maintaining picture quality. Other potential applications which come to mind, are patterns of weather balloon releases or other weather measurements, and patterms of seismic soundings, and ocean depth measurements.

The designs we have found take the following form:

Let $T_n^{j,k}$ be the tensor product design $\{\frac{\xi}{n^j}, \frac{\eta}{n^k}\}_{\xi=1,\eta=1}^{n^j \quad n^k}$ defined in (1.3) and let

$$T_{N,\ell} = \bigcup_{j=1}^{\ell+1} T_n^{j,\ell+2-j}, \qquad \ell = 0,1,2,... \tag{1.6}$$

The $\ell = 0$ and $\ell = 1$ designs have already been described. Figure 1.3 shows $T_n^{1,3}$, $T_n^{2,2}$, $T_n^{3,1}$ and $T_{N,2}$ for $n = 3$, $N = 189$.

It is easy to see that

$$T_n^{j,k+r} \cap T_n^{j+s,k} = T_n^{j,k},$$

for r,s non-negative integers, and it can be seen, with reference to the Venn diagram of Figure 1.4a, that

$$T_{N,\ell} = \sum_{j=1}^{\ell+1} T_n^{j,\ell+2-j} - \sum_{j=1}^{\ell} T_n^{j,\ell+1-j} . \tag{1.7}$$

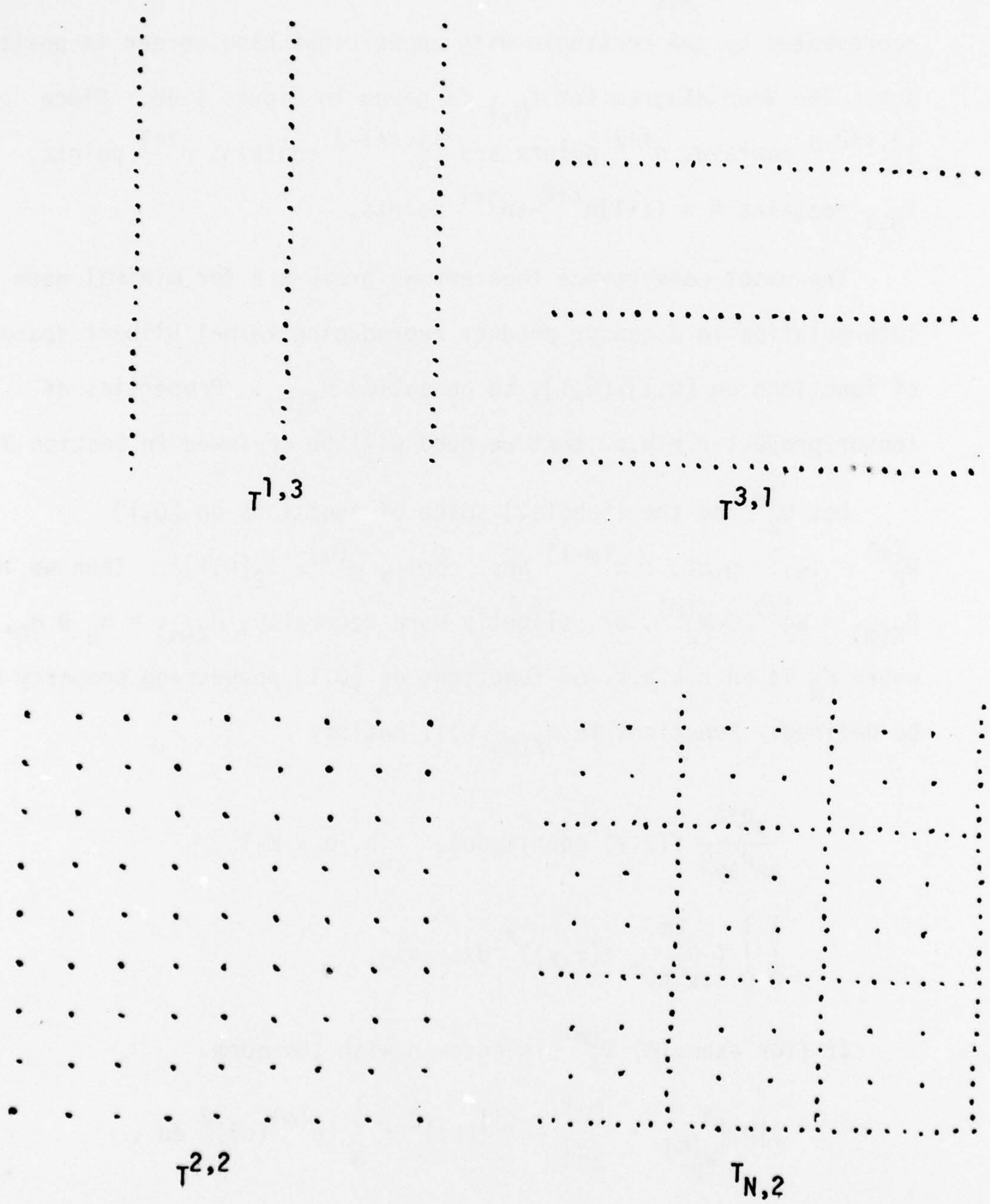


Figure 1.3

The designs $T_n^{1,3}$, $T_N^{2,2}$, $T_n^{3,1}$ and $T_{N,\ell}$ for $n=3$, $\ell=2$, $N=189$

In Figure 1.4, $T_{N,\ell}$ is outlined boldly, and each set $T_n^{j,k}$ is represented by the rectangle with upper right hand corner in position j,k. The Venn diagram for $T_{N,1}$ is given in Figure 1.4b. Since $T_n^{j,\ell+2-j}$ contains $n^{\ell+2}$ points and $T_n^{j,\ell+1-j}$ contains $n^{\ell+1}$ points, $T_{N,\ell}$ contains $N = (\ell+1)n^{\ell+2}-\ell n^{\ell+1}$ points.

The exact convergence theorems we prove are for minimal norm interpolation in a tensor product reproducing kernel Hilbert space of functions on $[0,1]\times[0,1]$, to be called $H_{K(m)}$. Properties of tensor product r.k.h.s. that we need will be reviewed in Section 3.

Let $W_2^{(m)}$ be the (Sobolev) space of functions on $[0,1]$: $W_2^{(m)} = \{g,: g,g',\ldots,g^{(m-1)}$ abs. cont., $g^{(m)} \varepsilon L_2[0,1]\}$. Then we let $H_{K(m)} = W_2^{(m)} \otimes W_2^{(m)}$, or, slightly more generally, $H_{K(m)} = H_Q \otimes H_Q$, where H_Q is an r.k.h.s. of functions on $[0,1]$ possessing property m to be defined. Functions in $H_{K(m)}$ will satisfy

$$\frac{\partial^{p+q}}{\partial x^p \partial y^q} f(x,y) \text{ continuous}, \qquad p,\ q \leq m-1$$

$$\int_0^1 \int_0^1 \left(\frac{\partial^{2m}}{\partial x^m \partial y^m} f(x,y)\right)^2 dxdy < \infty.$$

If (for example) $W_2^{(m)}$ is endowed with the norm

$$||g||^2_{W_2^{(m)}} = \sum_{j=0}^{m-1} (g^{(j)}(0))^2 + \int_0^1 (g^{(m)}(u))^2\, du\ , \tag{1.8}$$

then minimal norm interpolation in $W_2^{(m)}$ is polynomial spline interpolation of degree 2m-1 (see [14]) and minimal norm interpolation in $W_2^{(m)} \otimes W_2^{(m)}$ can be shown to be a form of bi-(2m-1)-ic polynomial spline interpolation. We argue that our convergence results must

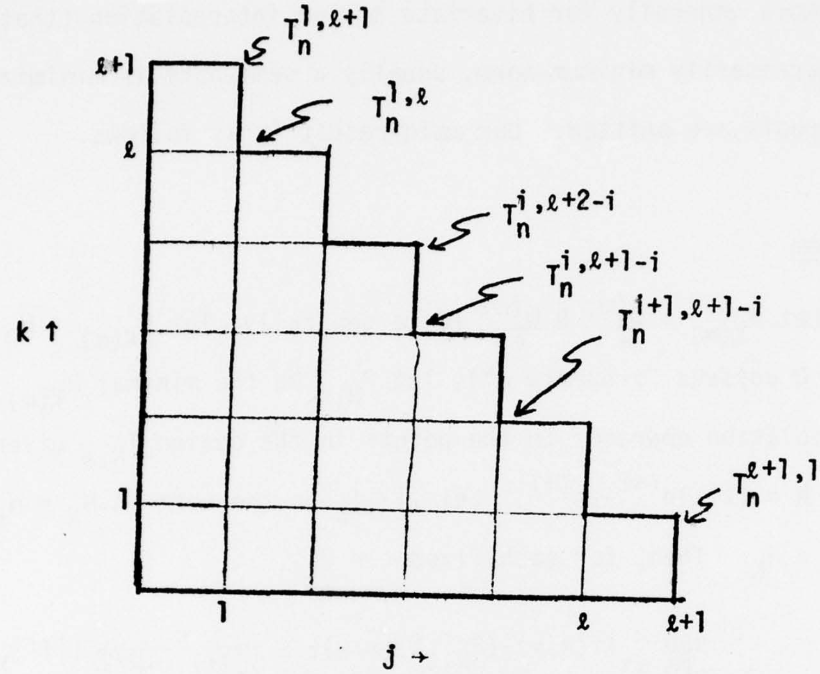


Figure 1.4(a)

Venn Diagram For $T_{N,\ell}$

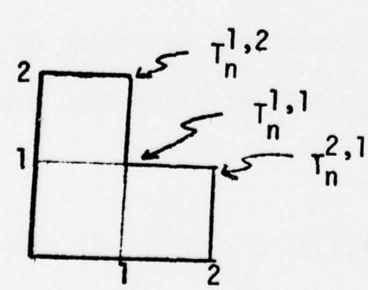


Figure 1.4(b)

Venn Diagram for $T_{N,1}$

hold more generally for bivariate spline interpolation (that is, not necessarily minimum-norm, usually a semi-norm is minimized) but the proofs are omitted. Our main result is as follows.

Theorem

Let $H_{K(m)} = W_2^{(m)} \otimes W_2^{(m)}$ (more generally, let $H_{K(m)} = H_Q \otimes H_Q$ where Q possess "property m"), let $P_{N,\ell}$ be the minimal $H_{K(m)}$ norm interpolation operator to the points in the design $T_{N,\ell}$ given by (1.6), hence $N = (\ell+1)n^{\ell+2} - \ell n^{\ell+1}$. Let $||\cdot||_K$ be the norm in $H_K = H_{K(m)}$ and let $f \in H_K$. Then, for each fixed $\ell = 0,1,\ldots,$

$$\sup_{x,y\in[0,1]} |f(x,y)-(P_{N,\ell}f)(x,y)| \le \frac{c}{(n^{\ell+1})^{m-(1/2)}} ||f||_K (2d+\ell c+\frac{(\ell+1)c}{n}) \tag{1.9}$$

$$= O[(\ell+2)(\frac{\ell+1}{N})^{\frac{\ell+1}{\ell+2}(m-(1/2))}] , \tag{1.10}$$

as $n \to \infty$

where c and d are two constants depending on Q. If, (loosely) furthermore $f \in H_{K(2m)}$ and satisfies some boundary conditions, then as $n \to \infty$,

$$\sup_{x,y\in[0,1]} |f(x,y)-(P_{N,\ell}f)(x,y)| = O((\ell+2)^2(\frac{\ell+1}{N})^{\frac{\ell+1}{\ell+2}(2m-(1/2))}). \tag{1.11}$$

The $\ell = 0$ results correspond to known results for interpolation on tensor product grids, in particular (1.11) corresponds closely to Mansfield [17], equation (4.2).

To get a feel for numbers, consider a typical tensor product grid of $N = n\times n = 200\times 200 = 40,000$ such as might occur in reconstruction

of the internal structure of the human body from multiple X-ray projections. X-ray dosage is proportional to N. (See Section 4 for further details). In the application described in Section 4 it is argued that $m = 1$ is appropriate, (1.9) holds and $c = d$.

Letting $c = d$, $\theta = c^2||f||_K$ in (1.9), the error bound is $\frac{\theta}{n^{(\ell+1)/2}}(\ell+2+\frac{\ell+1}{n})$, $\ell = 0,1,\ldots$, we obtain the error bound for the tensor product grid $T_{N,0}$ as

$$\frac{\theta}{200^{1/2}}(2+\frac{1}{200}) = .142\theta \ .$$

Consider alternatively the design T_N^ℓ with $n = 10$, $\ell = 2$, $N = (\ell+1)n^{\ell+2}-\ell n^{\ell+1} = 28{,}000$. The error bound becomes

$$\frac{\theta}{1{,}000^{1/2}}(4+\frac{3}{10}) = .136\theta \ .$$

Thus, one can maintain about the same error bound with a 30% reduction in N. One of the Gordon designs $T_{N,1}$ is better than the tensor product design for example $\ell = 1$, $n = 27$ gives $N = 39{,}366$, with a reduction of the error bound to $.114\theta$.

In applications where the function to be reconstructed has a greater degree of smoothness, (i.e. $m > 1$) the relative improvement can be very much greater.

In Section 2, we discuss interpolation operators of a fairly general class, and obtain formulas for the remainder operator $I - P_{N,\ell}$, where $P_{N,\ell}$ is any interpolation operator of a general class, not necessarily minimum norm. In Section 3, we review certain properties of tensor product r.k.h.s. and prove (1.9), (1.10) and (1.11). In Section 4,

we suggest the possible application of the results to reducing the X-ray dosage in computerized image reconstruction. In Section 5 we mention another potential application and some related unsolved problems, and in Section 6, we indicate how certain design problems for interpolation are equivalent to design problems for indirect sensing experiments.

2. Interpolation Operators, Remainder Formulae

In this section we develop properties of a class of interpolation operators. The results extend those of Gordon [8,9].

Let $\{\phi_j^M\}_{j=1}^M$ be any M real valued functions on [0,1] with the property

$$\phi_j^M(\tfrac{i}{M}) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \qquad i,j = 1,2,\ldots,M \tag{2.1}$$

For f a real-valued function on $[0,1]\times[0,1]$, define the operators $P_x^{N_1}$, and $P_y^{N_2}$ by

$$(P_x^{N_1}f)(x,y) = \sum_{i=1}^{N_1} \phi_i^{N_1}(x) f(\tfrac{i}{N_1},y) \tag{2.2}$$

$$(P_y^{N_2}f)(x,y) = \sum_{j=1}^{N_2} \phi_j^{N_2}(y) f(x,\tfrac{j}{N_2}) \tag{2.3}$$

and

$$(P^{N_1,N_2}f)(x,y) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \phi_i^{N_1}(x)\phi_j^{N_2}(y) f(\tfrac{i}{N_1},\tfrac{j}{N_2}) . \tag{2.4}$$

It is obvious that

$$P_x^{N_1} P_y^{N_2} = P_y^{N_2} P_x^{N_1} = P^{N_1,N_2} , \tag{2.5}$$

and that $P^{N_1,N_2}f$ interpolates to f for all (x,y) on the tensor product design $\{\frac{i}{N_1}, \frac{j}{N_2}\}_{i=1,j=1}^{N_1 \;\; N_2}$.

We will use the notation

$$P^{n^j,n^k} \equiv P_n^{j,k} ,$$

thus $P_n^{j,k}f$ interpolates to f for all (x,y) in the set $T_n^{j,k}$ defined in (1.3). Define

$$P_{N,\ell} = \sum_{j=1}^{\ell+1} P_n^{j,\ell+2-j} - \sum_{j=1}^{\ell} P_n^{j,\ell+1-j} . \qquad (2.6a)$$

where

$$N = (\ell+1)n^{\ell+2} - \ell n^{\ell+1} . \qquad (2.6b)$$

$P_{N,\ell}f$ depends on f only through its values (x,y) in the set $T_{N,\ell}$ defined in (1.7).

Terms in (2.6a) look, for example, like

$$(P_n^{j,\ell+2-j}f)(x,y) = \sum_{i=1}^{n^j} \sum_{k=1}^{n^{\ell+2-j}} \phi_i^{n^j}(x)\phi_k^{n^{\ell+2-j}} f\left(\frac{i}{n^j}, \frac{k}{n^{\ell+2-j}}\right) \qquad (2.6c)$$

We have the following

<u>Theorem 2.1</u>

$P_{N,\ell}f$ interpolates to f at all $(x,y) \in T_{N,\ell}$.

<u>Proof</u>:

Denote (x,y) by $\underset{\sim}{t}$. If $\underset{\sim}{t} \in T_n^{j,\ell+2-j}$ for all $j = 1,2,\ldots,\ell+1$, then $\underset{\sim}{t} \in T_n^{j,\ell+1-j}$ for $j = 1,2,\ldots,\ell$, each $P_n^{j,k}$ on the right of (2.6a) interpolates to f at $\underset{\sim}{t}$ and so $(P_{N,\ell}f)(\underset{\sim}{t}) = (\ell+1)f(\underset{\sim}{t}) - \ell f(\underset{\sim}{t}) = f(\underset{\sim}{t})$. In general, if $\underset{\sim}{t} \in T^{j,\ell+2-j}$ for $j = a$, $a+1, \ldots, b$, then $\underset{\sim}{t} \in T_n^{j,\ell+1-j}$ for $j = a$, $a+1, \ldots,$ $b-1$ and then

$$\sum_{j=a}^{b} (P_n^{j,\ell+2-j}f)(\underset{\sim}{t}) - \sum_{j=a}^{b-1} (P_n^{j,\ell+1-j}f)(\underset{\sim}{t}) = f(\underset{\sim}{t}).$$

Consider a $j = c$ for which $\underset{\sim}{t} \notin T^{c,\ell+2-c}$, and suppose, for concreteness, it is because x cannot be represented in the form $x = \frac{\xi}{n^c}$, for some $\xi = 1,2,\ldots,n^c$. Then there must exist $a > c$ such that $x = \frac{\xi_*}{n^a}$, and then there must exist n_* such that

$$y = \frac{n_*}{n^{\ell+2-a}} = \frac{n^{a-c}\, n_*}{n^{\ell+2-c}} = \frac{n^{a-c-1}\, n_*}{n^{\ell+1-c}} . \qquad (2.7)$$

Now

$$(P_n^{c,\ell+2-c}\ f)(x,y) - (P_n^{c,\ell+1-c}\ f)(x,y)$$

$$= \sum_{\xi=1}^{n^c} \phi_\xi^{n^c}(x) \Big\{ \sum_{\psi=1}^{n^{\ell+2-c}} \phi_\psi^{n^{\ell+2-c}}(y) f\left(\frac{\xi}{n^c}, \frac{\psi}{n^{\ell+2-c}}\right)$$

$$- \sum_{\rho=1}^{n^{\ell+1-c}} \phi_\rho^{n^{\ell+1-c}}(y) f\left(\frac{\xi}{n^c}, \frac{\rho}{n^{\ell+1-c}}\right)\Big\} \qquad (2.8)$$

Considering the two sums inside the brackets in (2.8) all the ϕ's are zero at y of (2.7) except when $\psi = n^{a-c}\ n_*$ and $\rho = n^{a-c-1}\ n_*$, and so the right hand side of (2.8) becomes

$$\sum_{\xi=1}^{n^c} \phi_\xi^{n^c}(x)\{f(\frac{\xi}{n^c}, y) - f(\frac{\xi}{n^c}, y)\} = 0$$

Thus

$$\sum_{j \ni \underset{\sim}{t} \notin T_n^{j,\ell+2-j}} ((P_n^{j,\ell+2-j} - P^{j,\ell+1-j})f)(x,y) = 0 .$$

A similar argument holds reversing the roles of x and y, and so the theorem is proved.

We will now develop a formula for the remainder operator $I - P_{N,\ell}$, in terms of the remainder operators $R_x^{N_1}$ and $R_y^{N_2}$ defined by

$$R_x^{N_1} \equiv I - P_x^{N_1}$$
$$R_y^{N_2} \equiv I - P_y^{N_2} \ . \tag{2.9}$$

<u>Lemma 2.1</u> (Gordon)

For any N_1 and N_2 we have the following decomposition of the identity:

$$I = P_x^{N_1} P_y^{N_2} + R_x^{N_1} + R_y^{N_2} - R_x^{N_1} R_y^{N_2} \tag{2.10}$$

<u>Proof</u>:

The right hand side of (2.10) equals

$$P_x^{N_1} P_y^{N_2} + (I - P_x^{N_1}) + (I - P_y^{N_2}) - (I - P_x^{N_1})(I - P_y^{N_2})$$

$$= P_x^{N_1} P_y^{N_2} + 2I - P_x^{N_1} - P_y^{N_2} - (I - P_x^{N_1} - P_y^{N_2} + P_x^{N_1} P_y^{N_2}) \ .$$

The formula for the remainder is given by

<u>Theorem 2.2</u>

$$I - P_{N,\ell} \equiv R_x^{n^{\ell+1}} + R_y^{n^{\ell+1}} + \sum_{j=1}^{\ell} R_x^{n^j} R_y^{n^{\ell+1-j}} - \sum_{j=1}^{\ell+1} R_x^{n^j} R_y^{n^{\ell+2-j}} \tag{2.11}$$

<u>Proof</u>:

By the definitions (2.6) and (2.2-2.4)

$$P_{N,\ell} = \sum_{j=1}^{\ell+1} P_x^{n^j} P_y^{n^{\ell+2-j}} - \sum_{j=1}^{\ell} P_x^{n^j} P_y^{n^{\ell+1-j}} \ . \tag{2.12}$$

By (2.9)

$$I = P_x^{n^j} P_y^{n^{\ell+2-j}} + R_x^{n^j} + R_y^{n^{\ell+2-j}} - R_x^{n^j} R_y^{n^{\ell+2-j}} \qquad j = 1,2,\ldots,\ell+1 \tag{2.13,j}$$

$$I = P_x^{n^j} P_y^{n^{\ell+1-j}} + R_x^{n^j} + R_y^{n^{\ell+1-j}} - R_x^{n^j} R_y^{n^{\ell+1-j}} \qquad j = 1,2,\ldots,\ell \ . \tag{2.14,j}$$

If we sum equations (2.13,j) over $j = 1,2,\ldots,\ell+1$ and subtract the sum of equations (2.14,j) over $j = 1,2,\ldots,\ell$ all terms of the form $R_z^{n^j}$ for $j \neq \ell+1$ cancel, and we obtain

$$(\ell+1)I - \ell I = P_{N,\ell} + R_x^{n^{\ell+1}} + R_y^{n^{\ell+1}} + \sum_{j=1}^{\ell} R_x^{n^j} R_y^{n^{\ell+1-j}} - \sum_{j=1}^{\ell+1} R_x^{n^j} R_y^{n^{\ell+2-j}} \ ,$$

giving the theorem.

The pleasant thing about the remainder operators on the right of (2.11) is that they will turn out to have error bounds which behave like those for interpolation over meshes of fineness $1/n^{\ell+1}$.

3. <u>Interpolation in Reproducing Kernel Hilbert Spaces</u>

The basic reference on r.k.h.s. that we use is Aronszajn [1], and assertions concerning r.k.h.s. not proved here may be found there. Let T be any index set and H a Hilbert space of functions on T. H is a (real) r.k.h.s. if and only if, for every $\tau \in T$, the linear functional $L_\tau h \to h(\tau)$ is continuous. Then, by the Riesz representation theorem, there exists an element, say $R_\tau \in H$ such that

$$h \in H \Rightarrow \langle h, R_\tau \rangle = h(\tau) \ . \tag{3.1}$$

The reproducing kernel (r.k.) $R(\sigma,\tau)$, σ, $\tau \in T$, for the space H, is then defined by

$$R(\sigma,\tau) = \langle R_\sigma, R_\tau \rangle .$$

We will henceforth denote by H_R the r.k.h.s. with r.k. $R(\cdot,\cdot)$, and the inner product in H_R by $\langle\cdot,\cdot\rangle_R$.

An r.k. is always non-negative definite, that is

$$\sum_{j,k=1}^{M} a_j a_k R(\sigma_j,\sigma_k) = \left|\left| \sum_{j=1}^{M} a_j R_{\sigma_j} \right|\right|_R^2 \geq 0 , \tag{3.3}$$

and, in this paper for any r.k. we shall always assume strict inequality on the right hand side of (3.3).

Reproducing kernel spaces are exactly those Hilbert spaces for which norm convergence implies pointwise convergence, in particular, if P is any linear operator from $H_R \to H_R$, then

$$|h(\tau)-(Ph)(\tau)| = |\langle h-Ph, R_\tau\rangle_R| \leq ||(I-P)h||_R ||R_\tau||_R .$$

If P is idempotent with adjoint P* (that is, P satisfies $P = P^2$, and P* satisfies $\langle h_1, Ph_2\rangle_R \equiv \langle P^*h_1, h_2\rangle_R$, all h_1, $h_2 \in H_R$) then

$$|h(\tau)-(Ph)(\tau)| = |\langle h-Ph, R_\tau - P^*R_\tau\rangle_R| \leq ||(I-P)h||_R ||(I-P^*)R_\tau||_R . \tag{3.4}$$

Given any strictly positive definite kernel, $R(\sigma,\tau)$, H_R exists and can be constructed as the closure of the span of $\{R_\tau, \tau \in T\}$ with the norm (3.3).

Let H_{Q_1} and H_{Q_2} be two r.k.h.s. of functions defined on the sets X and Y, respectively. Then the tensor product space H_K, say, defined by

$$H_K = H_{Q_1} \otimes H_{Q_2}$$

is a Hilbert space of functions on $T = X \otimes Y$ of the form

$$f(x,y) = \sum_{k,\ell=1}^{\infty} a_{k\ell}\, \psi_{1k}(x)\psi_{2\ell}(y), \qquad x \in X,\ y \in Y ,$$

where the $\{\psi_{ik}\}_{k=1}^{\infty}$ are a complete orthonormal sequence in H_{Q_i}, and $\sum_{k,\ell} |a_{k\ell}|^2 < \infty$. (See [1]).

If

$$\begin{aligned} f(x,y) &= \sum_k f_{1,k}(x) f_{2,k}(y) \\ g(x,y) &= \sum_\ell g_{1,\ell}(x) g_{2,\ell}(y) \end{aligned} \tag{3.5}$$

where the f_{ik}, g_{ik} are all in H_{Q_i}, then

$$\langle f,g\rangle_K = \sum_{k,\ell}\sum \langle f_{1,k}, g_{1,\ell}\rangle_{Q_1} \langle f_{2,k}, g_{2,\ell}\rangle_{Q_2} . \tag{3.6}$$

Although the representations (3.5) are not unique, the inner product (3.6) is. Letting $\underset{\sim}{s} = (u,v)$, $\underset{\sim}{t} = (x,y)$, r.k.'s for H_K and H_Q are related by

$$K(\underset{\sim}{s},\underset{\sim}{t}) \equiv K(u,v;x,y) = Q_1(u,x)Q_2(v,y) ,$$

equivalently

$$K_{\underset{\sim}{t}}(\cdot) = Q_{1x}(\cdot)Q_{2y}(\cdot) , \quad \text{where } Q_{iz}(\cdot) \equiv Q_i(\cdot,z) \tag{3.7}$$

Some examples of tensor product r.k.h.s. may be found in [16].

In the remainder of this paper we let $X = Y = [0,1]$, hence $T = [0,1]\times[0,1]$, and, without any real loss of generality, let $H_{Q_1} = H_{Q_2} = H_Q$. Let A_M be the operator in H_Q defined by

$$(A_M g)(z) = \sum_{j=1}^{M} \phi_j^M(z) g(\tfrac{j}{M}), \qquad z \in [0,1]. \tag{3.8}$$

where we suppose $\{\phi_j^M\}_{j=1}^M$ are as in Section 2 and are also in H_Q, and let A_M^* be the H_Q - adjoint of A_M. It is easy to verify that

$$A_M^* g = \sum_{j=1}^{M} \langle \phi_j^M, g \rangle_Q \, Q_{\frac{j}{M}} \, ,$$

where $Q_{\frac{j}{M}}(z) = Q(\frac{j}{M}, z)$ since

$$\langle f, A_M^* g \rangle_Q = \sum_{j=1}^{M} f(\tfrac{j}{M}) \langle \phi_j^M, g \rangle_Q = \langle A_M f, g \rangle_Q \, .$$

Lemma 3.1

Suppose the $\{\phi_j^M\}$ are in H_Q, and let the remainder operators $R_x^{N_1}$ and $R_y^{N_2}$ be defined by (2.9). Then the H_K adjoints $R_x^{N_1 *}$ and $R_y^{N_2 *}$ of $R_x^{N_1}$ and $R_y^{N_2}$ are well defined and satisfy

$$||R_x^{N_1 *} K_{\underset{\sim}{t}}||_K = ||(I-A_{N_1}^*)Q_x||_Q ||Q_y||_Q$$

$$||R_y^{N_2 *} K_{\underset{\sim}{t}}||_K = ||(I-A_{N_2}^*)Q_y||_Q ||Q_x||_Q$$

$$||R_x^{N_1 *} R_y^{N_2 *} K_{\underset{\sim}{t}}||_K = ||(I-A_{N_1}^*)Q_x||_Q ||(I-A_{N_2}^*)Q_y||_Q \, .$$

Proof:

The proof proceeds by verifying that if $f(x,y) \in H_K$ has the representation

$$f(x,y) = \sum_{k=1}^{\infty} f_{1k}(x) f_{2k}(y)$$

where the $\{f_{ik}\}$ are in H_Q, then $P_x^{N_1 *}$, defined by

$$(P_x^{N_1 *} f)(x,y) = \sum_{k=1}^{\infty} \sum_{j=1}^{N_1} Q_{\frac{j}{N_1}}(x) \langle f_{1k}, \phi_j^{N_1} \rangle_Q f_{2k}(y)$$

is the H_K adjoint to $P_x^{N_1}$. Then

$$R_x^{N_1 *} K_{\underset{\sim}{t}} = (I-P_x^{N_1 *}) K_{\underset{\sim}{t}} = Q_x Q_y - \sum_{j=1}^{N_1} Q_{\frac{j}{N_1}} \langle Q_x, \phi_j^{N_1} \rangle_Q Q_y$$

$$= (I-A_{N_1}^*) Q_x \cdot Q_y$$

and so, by (3.6)

$$||R_x^{N_1 *} K_{\underset{\sim}{t}}||_K = ||(I-A_{N_1}^*)Q_x||_Q ||Q_y||_Q \, .$$

Similarly

$$R_y^{N_2 *} R_x^{N_1 *} K_{\underset{\sim}{t}}$$

$$= (I-P_y^{N_2 *})(I-A_{N_1}^*)Q_x \cdot Q_y$$

$$= (I-A_{N_1}^*)Q_x \cdot Q_y - (I-A_{N_1}^*)Q_x \cdot A_{N_2}^* Q_y$$

$$= (I-A_{N_1}^*)Q_x \cdot (I-A_{N_2}^*)Q_y \, .$$

Collecting results we have the following

Theorem 3.1

Suppose

$$\sup_z ||(I-A_M^*)Q_z||_Q \le \frac{c}{M^p}$$

for some c and p independent of M, and let $d = \sup_z ||Q_z||_Q$. Then, for $\ell = 0,1,2,\ldots,$

$$\sup_t |f(t)-(P_{N,\ell}f)(t)| \le ||(I-P_{N,\ell})f||_K \frac{c}{n^{(\ell+1)p}}[2d+\ell c+\frac{(\ell+1)}{n}c]$$

$$\le ||(I-P_{N,\ell})f||_K \max(c^2,cd)\cdot(\ell+2)(\frac{\ell+1}{N})^{\frac{\ell+1}{\ell+2}p}(1+o(1)).$$

Proof:

By (3.4), since $P_{N,\ell}$ is idempotent, we have

$$|f(\underset{\sim}{t})-(P_{N,\ell}f)(\underset{\sim}{t})| \le ||(I-P_{N,\ell})f||_K ||(I-P^*_{N,\ell})K_{\underset{\sim}{t}}||_K .$$

By Theorem 2.2

$$||(I-P^*_{N,\ell})K_{\underset{\sim}{t}}|| \le ||R_x^{n^{\ell+1}} * K_{\underset{\sim}{t}}||_K + ||R_y^{n^{\ell+1}} * K_{\underset{\sim}{t}}||_K$$

$$+ \sum_{j=1}^{\ell} ||R_x^{n^j} * R_y^{n^{\ell+1-j}} * K_{\underset{\sim}{t}}||_K$$

$$+ \sum_{j=1}^{\ell+1} ||R_x^{n^j} * R_y^{n^{\ell+2-j}} * K_{\underset{\sim}{t}}||_K$$

Substituting in the results of Lemma 3.1 and the hypotheses complete the proof.

Remark: Note that, provided $||(I-A_M)||_Q$ and hence $||I-P_{N,\ell}||_K$, are bounded, then this theorem relates one dimensional interpolation error rates to bivariate error rates using the design $T_{N,\ell}$, $\ell = 0,1,\ldots$, since, for $g \in H_Q$,

$$|g(z)-\sum_{j=1}^{M}\phi_j^M(z)g(\tfrac{j}{M})| = |\langle g-A_M g, Q_z\rangle_Q|$$

$$\le ||(I-A_M)g||_Q ||(I-A_M^*)Q_z||_Q .$$

Later we will discuss without proofs interpolation operators for which $||(I-P_{N,\ell})f||_K \le$ const. $||f||_K$. If, however $P_{N,\ell}$ is an orthogonal projector, then $||(I-P_{N,\ell})f||_K \le ||f||_K$.

Now choose for the $\{\phi_j^M\}$ the special functions defined by

$$(\phi_1^M,\phi_2^M,\ldots,\phi_M^M) = (Q_{\frac{1}{M}},Q_{\frac{2}{M}},\ldots,Q_{\frac{M}{M}})Q_M^{-1}, \tag{3.9}$$

where Q_M is the M×M matrix with jkth entry $Q(\frac{j}{M},\frac{k}{M})$.

We have the following

Lemma 3.2

Let the $\{\phi_j^M\}$ be defined as in (3.9). Then A_M is the orthogonal projector in H_Q onto span$\{Q_{\frac{i}{M}}\}_{i=1}^M$, and $P_{N,\ell}$ and each of the operators in the remainder formula (2.11) of Theorem 2.2 are orthogonal projectors in H_K. (Thus, $P_{N,\ell}$ is that element of minimal norm in H_K interpolating to f on $T_{N,\ell}$).

Proof:

We have

$$A_M g = (Q_{\frac{1}{M}},\ldots,Q_{\frac{M}{M}})Q_M^{-1}\begin{pmatrix} g(\frac{1}{M}) \\ \vdots \\ g(\frac{M}{M})\end{pmatrix} . \tag{3.10}$$

clearly $A_M g \in$ span$\{Q_{\frac{j}{M}}\}_{j=1}^M$. Noticing that $(Q_{\frac{1}{M}}(z),\ldots,Q_{\frac{M}{M}}(z))$ evaluated at $z = j/M$ is the jth row of Q_M, it follows that $\phi_i^M(\frac{j}{M})$ is 1 or 0 according as $i = j$ or not and that $A_M g$ interpolates to g at $z=j/M$, $j = 1,2,\ldots,M$. Therefore, $(I-A_M)g$ satisfies the "normal equations"

$$\langle (I-A_M)g, Q_z\rangle_Q = 0, \quad z = \frac{1}{M}, \frac{2}{M}, \ldots, \frac{M}{M} .$$

Hence A_M is the orthogonal projector onto $\mathrm{span}\{Q_{\frac{j}{M}}\}_{j=1}^{M}$. Similarly, $P_{N,\ell}$ is the orthogonal projector onto $\mathrm{span}\{K_{\underset{\sim}{t}},\ \underset{\sim}{t} \in T_{N,\ell}\}$ since it can be verified that $P_{N,\ell}f \in \mathrm{span}\{K_{\underset{\sim}{t}}, \underset{\sim}{t} \in T_{N,\ell}\}$, $f \in H_K$ and, since $P_{N,\ell}f$ interpolates to f for all $\underset{\sim}{t} \in T_{N,\ell}$ by Theorem 2.1, $f-P_{N,\ell}f$ satisfies the normal equations

$$\langle f-P_{N,\ell}f, K_{\underset{\sim}{t}}\rangle_K = 0\,, \qquad \underset{\sim}{t} \in T_{N,\ell}\,.$$

A similar argument holds for each operator of the form P^{N_1,N_2} in the definition of $P_{N,\ell}$ of (2.6a). It remains to show that $P_x^{N_1}$ and $P_y^{N_2}$ are orthogonal projectors. We show that $P_x^{N_1}$ is the orthogonal projector onto

$$\overline{\mathrm{span}}\{K_{\underset{\sim}{t}},\ \underset{\sim}{t} \in [\tfrac{i}{N_1}, y],\ i = 1,2,\ldots,N_1,\ \ y \in [0,1]\}\,. \tag{3.11}$$

Since $\overline{\mathrm{span}}\{Q_y, y \in [0,1]\}$ is H_Q, the space in (3.11) is

$$\overline{\mathrm{span}}\{Q_{\frac{i}{N_1}} \otimes H_Q,\ i = 1,2,\ldots,N_1\}\,. \tag{3.12}$$

It is clear that $P_x^{N_1}f$ is in the space in (3.11) and that the normal equations

$$\langle f-P_x^{N_1}f, K_{\underset{\sim}{t}}\rangle_K = 0, \qquad \underset{\sim}{t} \in [\tfrac{i}{N_1} \otimes [0,1]]_{i=1}^{N_1}$$

are satisfied, so the lemma is proved.

We note that $A_M = A_M^*$ here.

We say that a r.k. $Q(z,z')$ on $[0,1]\times[0,1]$ has property m if

(i) $\frac{\partial^r}{\partial z^r} Q(z,z')$ exists and is continuous on $[0,1]\times[0,1]$ for $z \neq z'$, $r = 0,1,2,\ldots,2m$, $\frac{\partial^r}{\partial z^r} Q(z,z')$ exists and is continuous on $[0,1]\times[0,1]$ for $r = 0,1,\ldots,2m-2$

(ii) $\lim\limits_{z\uparrow z'} \frac{\partial^{2m-1}}{\partial z^{2m-1}} Q(z,z')$ and

$\lim\limits_{z\downarrow z'} \frac{\partial^{2m-1}}{\partial z^{2m-1}} Q(z,z')$ exist and are bounded for all $z' \in [0,1]$.

<u>Example</u>: Let $H_Q = W_2^{(m)}$ with the norm given by (1.8). It can be verified [14] that the r.k. is

$$Q(x,y) = \sum_{j=0}^{m-1} \frac{x^j y^j}{(j!)^2} + \int_0^{\min(x,y)} \frac{(x-u)_+^{m-1}}{(m-1)!}\,\frac{(y-u)_+^{m-1}}{(m-1)!}\,du\,,$$

and that Q has property m. Furthermore $Q_x(u)$ is a polynomial spline of degree 2m-1 with the single knot at u=x, and possessing 2m-2 continuous derivations.

<u>Lemma 3.3</u>

If Q has property m and the $\{\phi_j^M\}$ are given by (3.9), then there exists a constant $c^2 = c^2(Q)$ such that

$$\sup_{z\in[0,1]} ||(I-A_M)Q_z||_Q^2 \le \frac{c^2}{M^{2m-1}}\,.$$

<u>Proof</u>:

This is equivalent to equation (2.36) in [23].

We have the following

<u>Theorem 3.2a</u>

Let Q have property m and suppose that the $\{\phi_j^M\}$ are given by (3.9). Then for each $\ell = 0,1,2,\ldots,$

$$\sup_t |f(t)-P_{N,\ell}f(t)| \le ||f||_K \frac{c}{(n^{\ell+1})^{m-(1/2)}}[2d+\ell c+\frac{\ell+1}{n}c]$$

$$\le \text{const. } (\ell+2)(\frac{\ell+1}{N})^{\frac{\ell+1}{\ell+2}(m-(1/2))}(1+o(1))$$

where c is as in Lemma 3.3, $d = \sup_z ||Q_z||_Q$, and $o(1) \to 0$ as $n \to \infty$.

Proof:

This is a consequence of Theorem 3.1, the fact that $P_{N,\ell}$ is an orthogonal projector, and Lemma 3.3.

If further "smoothness" conditions are assumed for f, beyond those entailed by $f \in H_K$, then higher order convergence rates can be obtained.

Lemma 3.4

Let f have a representation

$$f(x,y) = \int_0^1\int_0^1 K(x,y;u,v)\rho(u,v)dudv \qquad (3.13)$$

for some $\rho \in L_2([0,1]\times[0,1])$, and suppose Q has property m. Then there exist constants e_1 and e_2 such that

$$||R_x^{N_1}f||_K^2 \le \frac{e_1^2}{N_1^{2m}} \int_0^1\int_0^1 \rho^2(u,v)dudv \qquad (3.14)$$

and

$$||R_x^{N_1} R_y^{N_2}f||_K^2 \le \frac{e_2^2}{N_1^{2m}N_2^{2m}} \int_0^1\int_0^1 \rho^2(u,v)dudv. \qquad (3.15)$$

Proof:

The proof is a straightforward extension of the proof of the one dimensional analogue in [23] to the two dimensional case, and an outline of it is given in the appendix.

Remark: The condition (3.13) is equivalent to $f \in H_{K^2}$ where K^2 is the r.k.

$$K^2(x,y;u,v) = \iint_{\xi\eta} K(x,y;\xi,\eta)K(u,v;\xi,\eta)d\xi d\eta$$

see [18], furthermore $||f||_{K^2} = \int_0^1\int_0^1 \rho^2(u,v)dudv$. If $H_Q = W_2^{(m)}$ then $f \in H_{K^2}$ if $f \in W_2^{(2m)} \otimes W_2^{(2m)}$ and in addition, satisfies some boundary conditions. The condition $f \in H_{K^2}$ is reminiscent of the additional smoothness and boundary conditions one must impose in order to get high order convergence rates for spline interpolation, see e.g., [13].

Lemma 3.5

If $f \in H_{K^2}$, and the hypotheses of Theorem 3.2a are satisfied, then

$$||f-P_{N,\ell}f||_K \le \frac{2e_1+\ell e_2}{(n^{\ell+1})^m} \; ||f||_{K^2} (1+o(1))$$

Proof:

This is a consequence of Theorem 2.2, and Lemma 3.4.

Theorem 3.2b

Let the hypotheses of Theorem 3.2a be satisfied, and suppose that $f \in H_{K^2}$. Then

$$\sup_{t} |f(t)-(P_{N,\ell}f)(t)| \leq \frac{(2cd+c^2\ell)(2e_1+e_2\ell)}{(n^{\ell+1})^{(2m-1/2)}} ||f||_{K^2} (1+o(1))$$

$$\leq \mu(\ell+2)^2 \left(\frac{\ell+1}{N}\right)^{\frac{\ell+1}{\ell+2}(2m-1/2)} ||f||_{K^2} (1+o(1))$$

where $\mu = \max(cd, c^2, e_1, e_2)$.

Proof:

Immediate.

Remarks: It is common to take $H_Q = W_2^{(m)}$ and let $A_M g$ be that element in $W_2^{(m)}$ which minimizes the seminorm $[\int_0^1 (g^{(m)}(u))^2 du]^{1/2}$, subject to the interpolating conditions. Then A_M is idempotent but not self-adjoint. The $\{\phi_j^M\}$ are spline functions. Based on known results for spline interpolation (see e.g. [13,22]) it is fairly clear that the same convergence rates can be obtained for this case, we omit proofs. For computational purposes the representation (3.9) for the $\{\phi_j^M\}$ is not a good one - the $\{\phi_j^M\}$ are generally better represented for computational purposes in terms of B-splines, see Schoenberg [21], de Boor [3,4]. For a discussion of computer manipulation of tensor products see de Boor [5]. A program for computing a bivariate cubic spline interpolate on a tensor product grid is in the IMSL library [11].

4. A Potential Application to Reduction of X-Ray Dosage in Image Reconstruction.

Reconstruction of the internal structure of the human body from X-ray projections is now done routinely. See [7]. The X-ray dosage received by a patient is directly proportional to the number N of data points collected, so it is of benefit to design the data collection device (X-ray machine with arrays of pencil-beams) to minimize N consistent with required levels of image reconstruction. Although there are various ways the preceeding design results might apply to image reconstruction we describe a potential application with respect to the detector geometry and mathematical techniques described in some detail and evaluated numerically by Herman and Naparstek [10]. We take pains to note that this application is marked potential, because the final evaluation of any proposed porcedure in image reconstruction must be made with respect to realistic data, which may well have properties that are glossed over by the mathematical idealizations. Moreover, we are dealing here with error bounds, rather than error estimates.

The geometrical description, figures and inversion formula are obtained from [10]. Figure 4.1 describes an idealized X-ray source at position S. The source moves around point O at radius D, and makes an angle β with the y axis, $0 \leq \beta \leq 2\pi$. The source emits a fan of idealized rays, the σ^{th} ray making an angle σ with the S-O axis as shown in Figure 4.1, $-\delta \leq \sigma \leq \delta$. For each fixed β, the intensity of the σ^{th} attenuated X-ray beam is measured at the σ^{th} position along the detector strip.

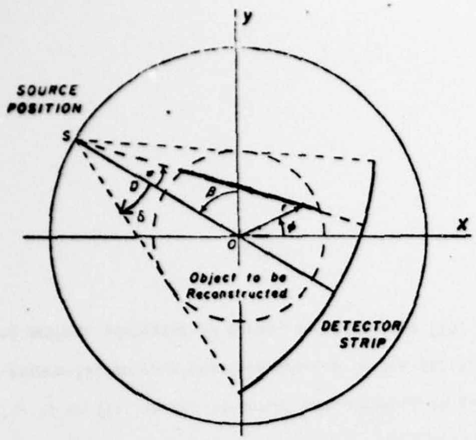


Figure 4.1

Geometry of Idealized Fan-Beam Data Collection

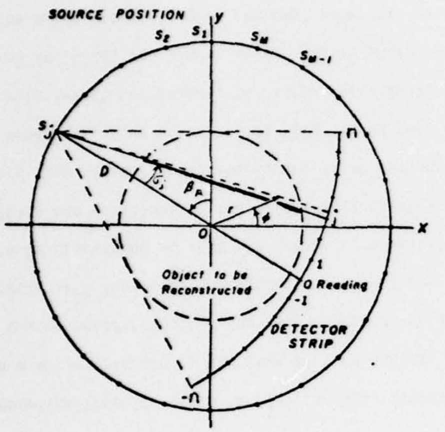


Figure 4.2

Geometry of Discretized Fan-Beam Data Collection

If $f(r,\phi)$ is the function to be reconstructed, the σ,βth "data point" $g(\sigma,\beta)$ is then the line integral of f along the σ^{th} ray while the source is in position β,

$$g(\sigma,\beta) = \int_{\sigma,\beta^{th}\ \text{line}} f(r,\phi)\, d\ell \tag{4.1}$$

Hermans and Naparstek [10] equation (10) provide the "Radon inversion formula for divergent beams" as

$$f(r,\phi) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\delta}^{\delta} \frac{1}{\sin(\sigma'-\sigma)} \frac{1}{U}\left[\frac{\partial g}{\partial \sigma}(\sigma,\beta) - \frac{\partial g}{\partial \beta}(\sigma,\beta)\right] d\sigma\, \partial\beta$$

where

$$U = \{[r\cos(\beta-\phi)]^2 + [D + r\sin(\beta-\phi)]^2\}^{1/2}$$

$$\sigma' = \tan^{-1} \frac{\cos(\beta-\phi)}{D + r\sin(\beta-\phi)} .$$

Hermans and Naparstek observe that the kernel $1/\sin(\sigma'-\sigma)$ is singular and propose replacing it by a regularizing kernel, which they call $Q_A(\sigma'-\sigma)$, where for each $A < \infty$, Q_A is nonsingular but for large A approximates $1/\sin(\sigma'-\sigma)$ in an appropriate sense. After some manipulation, their end result is an approximate inversion formula given by

$$\tilde{f}(r,\phi) = \frac{1}{4\pi^2} \int_0^{2\pi} \frac{D}{U^2} \int_{-\delta}^{\delta} H(\sigma'(r,\phi,\beta)-\sigma) g(\sigma,\beta)\, d\sigma\, d\beta \tag{4.2}$$

where $\tilde{f}$ is an approximation to f and $H = H_A$ has properties convenient for the evaluation of the integral by quadrature.

In the situation described by Hermans and Naparstek, the source is rotated through M equally spaced positions $S_1, S_2, \ldots, S_M$, and the idealized fan beam of width 2δ of Figure 4.1 actually consists of

$2n+1$ equally spaced rays making angles σ_j, $j = -n,\ldots,n$ with the line from source to origin. See Figure 4.2. Thus $g(\sigma,\beta)$ is observed on the tensor product design $(\sigma,\beta) = \{\sigma_j,\beta_k\}_{j=-n,k=1}^{n\ \ M}$ with $(2n+1)M = N$ points. Given this data, the integral (4.2) is evaluated by quadrature using the tropezoidal rule, first in σ, and using certain sepcial properties of H and linear interpolation, in β. Values of M and n are quite large, for example in their simulation of a cross section of the human thorax discussed in [10], $M = 288$ and $n = 150$, giving $N = 86,688$.

The theoretical results given earlier strongly suggest that a reduction in N while maintaining the same picture quality could be effected if g were measured not on the tensor product grid but on one of the better ℓ^{th} level designs suggested. The X-ray equipment would have to be redesigned so that the number of possible fan beams (presently $2n+1$) and positions of the detector (presently M) is increased, and then not all measurements and taken for each position. Hermans and Naparstek, (see also Rowland [19]) found that the tropezoidal rule (which corresponds to linear interpolation) for the inner integral in (4.2), and linear interpolation where necessary for evaluating the outer integral, gave the best numerical results among several methods tried. This suggests basing the evaluation of $\tilde{f}$ from $g(\sigma,\beta)$, $\sigma,\beta \in T_{N,\ell}$ on bilinear interpolation of g over the design set. This is equivalent to bivariate spline interpolation with m=1. One possibility is to fill in the "missing" measurements of $g(\sigma,\beta)$ by bilinear interpolation on g and then proceed to the parallel processing methods advocated by Herman and Naparstek [10] on p.523.

An analytical argument for the appropriateness of using bi-linear (i.e. m=1) interpolation on $g(\sigma,\beta)$ in this problem is as follows: Visualize a cross section of the (X-ray density of) the human brain. Pictures are given in [10]. The cross section of a tumor in an otherwise homogenous slice might be reasonably modeled as a function $f_0(r,\theta)$ which is, after normalization equal to 1 on an ellipse, and 0 in the remainder of the region to be reconstructed. See Figure 4.3. The dark area has density 1. What properties does the idealized function $g_0(\sigma,\beta)$ related to f_0 by (4.1) have? For β fixed, as σ sweeps from $-\delta$ to δ the line integral measures the presented cross section of the ellipse with respect to the σ^{th} ray and can be seen to be a continuous function of σ, but $\frac{\partial}{\partial\sigma} g(\sigma,\beta)$ will be discontinuous at σ_1 and σ_2. Similarly, for $\sigma = \sigma_k$, as the source revolves, $g(\sigma_k,\beta)$ will be a continuous function of β, but, depending on the position of the tumor and σ_k, $\frac{\partial g}{\partial\beta}(\sigma_k,\beta)$ may have discontinuities. Now $g(\sigma_k,\beta)$ is a periodic function of β, and $g(-\delta,\beta) = g(\delta,\beta) = 0$. In the β, σ rectangle, then one should take $H_Q = H_{Q_1}$ in the σ-direction

$$H_{Q_1} = \{h = h(\sigma):\ h \text{ abs. cont.},$$

$$h(-\delta) = h(\delta) = 0,\quad h' \in L_2[-\delta,\delta]\}\ ,$$

and $H_Q = H_{Q_2}$ in the β-direction as

$$H_{Q_2} = \{h = h(\beta):\ h \text{ abs. cont.},$$

$$h(0) = h(2\pi),\ h' \in L_2[0,2\pi]\}$$

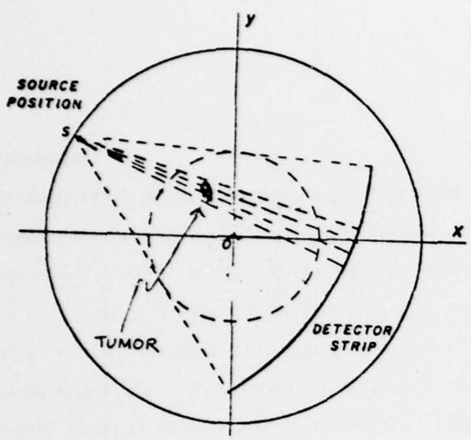


Figure 4.3

Cross section of a Tumor

The r.k. for $H_Q = H_{Q_1}$ with $[-\delta,\delta]$ transformed to $[0,1]$, and with $||h||^2_{Q_1} = \int_0^1 (h'(\sigma))^2 d\sigma$ is $Q_1(u,v) = \min(u,v) - uv$. To compare error bounds numerically for different ε as was done in the introduction, one needs to know c/d. Clearly $d^2 = \max_z Q(z,z)$ equals $\max_z z(1-z) = \frac{1}{4} \max_z z(1-z) = 1/4$. The quantity $\sup_z ||(I-A_M)Q_z||^2_Q$ is evaluated by observing that since minimal norm interpolation in this space is linear interpolation, if $z \in [\frac{i}{M}, \frac{i+1}{M}]$ then

$$A_M Q_z = M(\frac{i+1}{M} - z)Q_{\frac{i}{M}} + M(z - \frac{i}{M})Q_{\frac{i+1}{M}}$$

and

$$||(I-A_M)Q_z||^2_Q = \int_0^1 \{\frac{d}{du}[Q_z(u) - (i+1-Mz)Q_{\frac{i}{M}}(u) - (Mz-i)Q_{\frac{i+1}{M}}(u)]\}^2 du .$$

The integrand is 0 outside $[\frac{i}{M}, \frac{i+1}{M}]$, and carrying out the calculations at $z = \frac{i+1/2}{M}$, which will give the maximum, gives

$$\sup_z ||(I-A_M)Q_z||^2_Q = \frac{1}{4}\frac{1}{M} .$$

Thus c = d in this example. There is some arbitrariness in choosing the r.k. for H_{Q_2} which will affect c/d, however, using periodic linear interpolation, the error properties in practice should not be seriously altered.

We would like to thank Frank Natterer for an interesting discussion concerning the choice of appropriate spaces for X-ray reconstruction.

5. Other Applications, Unsolved Problems

A general design problem in an r.k.h.s. H_R of functions on T, may be formulated as follows:

Let T_N be an N-point subset of T. Given $f \in H_R$, find T_N to minimize

$$||f-P_N f||_R^2 , \tag{5.1}$$

where P_N is the minimal norm interpolation operator.
More generally, one wants (5.1) to be small over a class of f's. Rather detailed results on optimal T_N for $T = [0,1]$ and $f \in H_{R^2}$ have been given by Sacks and Ylvisaker, Wahba and others, see [20,24] and references cited there. Those results describe optimum designs in terms of an asymptotic (large N) point density on [0,1] but it can be seen from the results here (see also the remarks in [28]) that the description of a design on the unit square by a local point density is inadequate.

An important variation on this problem concerns the case where $f(\tau)$ is observed with error, i.e. the data are $y_\tau = f(\tau) + \epsilon_\tau$, where $\tau \in T_N$ and ϵ_τ is a random variable, say, with $E\,\epsilon_\tau = 0$, $E\,\epsilon_\sigma \epsilon_\tau = s$, $\sigma = \tau$, $= 0$, $\sigma \neq \tau$. Then f is approximated by a surface $\hat{f}$ which <u>smooths</u> the data $\{y_\tau, \tau \in T_N\}$, see [2,26] for details. A good design might be chosen to minimize $E||f-\hat{f}||_R^2$, say. This problem is formulated mathematically in [25] for the $T = [0,1]$ case but no solution is presented.

Numerous other practical problems can be formulated mathematically in terms of design problems like the one here. Another example is the location of air pollution measuring devices throughout a region, when the goal is to make a map of the air pollution levels throughout the region. Then one wants the design to be good for a class of realistic f's and the norm and the smoothing procedure (if any) must be chosen with respect to realistic terrain variations. The relation between prior covariance, and r.k.'s (see [14,26,27]) may be useful.

6. Applications to the design of indirect sensing experiments

In this section we show how design problems for indirect sensing experiments are mathematically equivalent to design problems for interpolation and smoothing.

The general design problem for indirect sensing experiments can be formulated as follows. Let

$$g(t) = \iint_S B(t,u) f(u)\, du , \qquad t \in T \tag{6.1}$$

where $K(\cdot,\cdot)$ is known and S, T are index sets, which we take as rectangles here. The function g will be observed for $t \in T_N \subset T$ and it is desired to choose T_N so that the estimate f_N of f based on $g(t)$, $t \in T_N$ is best in some sense. We shall show how the results of this paper can apply to the design problem for certain indirect sensing experiments.

Suppose $f \in H_R$ where H_R is a given r.k.h.s. of functions on S, with r.k. $R(u,v)$, u, $v \in S$, and, suppose for every $t \in T$, $|(Bf)(t)| \leq ||f||_R$. Then, by the Reisz representation theorem, there exists $\eta_t \in H_R$ such that

$$(Bf)(t) = \langle \eta_t, f \rangle_R .$$

Let

$$K(\underset{\sim}{s},\underset{\sim}{t}) = \langle n_s, n_t \rangle_R = \iint_S \iint_S B(\underset{\sim}{s},\underset{\sim}{u})R(\underset{\sim}{u},\underset{\sim}{v})B(\underset{\sim}{t},\underset{\sim}{v})d\underset{\sim}{u}\,d\underset{\sim}{v}$$

Then, letting V be the H_R-closure of the span of n_t, $t \in T$, there is an isometric isomorphism between $V \subset H_R$ and H_K, the r.k.h.s. with r.k. K generated by the correspondence "$\sim$",

$$n_t \in H_R \sim K_t \in H_K$$

where $K_t(s) = K(t,s)$. To see this, note that H_K is the closure of the span of $\{K_t, t \in T\}$ and

$$\langle n_t, n_s \rangle_R = K(t,s) = \langle K_t, K_s \rangle_K .$$

More details may be found in [23].

Since

$$(Bn_t)(s) = \langle n_s, n_t \rangle_R = K_t(s) ,$$

the correspondence

$$n_t \sim K_t , \qquad t \in T$$

is equivalent to

$$f \in V \sim g = Bf \in H_K ,$$

and it follows that $B(V) = H_K$. $V^{\perp}$ is the null space of B, since $\langle n_t, f \rangle_R = 0$, $t \in T \iff f$ in the null space of B.

Now, let $B_N f = ((Bf)(t_1),\ldots,(Bf)(t_N))'$, B_N: $H_R \to E_N$, and for any $g_N \in B_N(H_R)$ let $B_N^{+} g_N = f_N$, be that element in H_R of minimal norm satisfying $B_N f_N = g_N$; B_N^{+}: $E_N \to V$. Given data

$g_N = (g(t_1),\ldots,g(t_N))'$, we let the approximate solution to (6.1) be $f_N = B_N^{+} g_N$.

We consider as a design criteria the choice of $t_1,\ldots,t_N$ to minimize, say, $||f-f_N||_R^2$. Now, for $g \in H_K = B(H_R)$, let $B^{+}g$ be that element of minimal H_R norm satisfying $BB^{+}g = g$. Then

$$||f-f_N||_R^2 = ||f-B_N^{+}B_N f||_R^2 = ||f-B^{+}Bf||_R^2 + ||B^{+}Bf-f_N||_R^2 .$$

Since $f-B^{+}Bf$ is in the null space of B, no information about $f-B^{+}Bf$ is obtained from observing Bf, and the magnitude of $||f-B^{+}Bf||_R^2$ is irrelevant to the design problem. Thus, we consider the choice of $t_1,\ldots,t_N$ to minimize $||B^{+}Bf-f_N||_R^2$. This may be reduced to a design problem for interpolation by noting that if $g = Bf$, and $P_N g$ is that element of minimal H_K norm interpolating to g at $t = t_1,\ldots,t_N$, then

$$B^{+}Bf \sim g$$

$$f_N \sim P_N g$$

and so

$$||B^{+}Bf-f_N||_R^2 = ||g-P_N g||_K$$

Thus, choosing $t_1,\ldots,t_N$ so that $P_N g$ is a good interpolant to g, in the H_K norm, is equivalent to choosing $t_1,\ldots,t_N$ so that f_N is close to $B^{+}Bf$ in the H_R-norm.

If H_R is a tensor product space, and $B(t,u) = B_1(x,\xi)B_2(y,\eta)$ say, where $t = (x,y)$, $u = (\xi,\eta)$, then H_K will be a tensor product space.

In practice, one might prefer to assume that one observes $y(t) = g(t)+\epsilon(t)$, $t = t_1,\ldots,t_N$, where $\epsilon(t)$ is a random variable with,

for example $E\,\epsilon(t) = 0$, $E\,\epsilon(s)\epsilon(t) = \sigma^2$, $s = t$, $= 0$ otherwise. This model leads to a regularized estimate for f (see e.g. [27]) whereas the calculation of f_N will in general be an ill conditioned problem and can give nonsense answers if the data has errors. Let $f_{N,\lambda}$ be the minimizer in H_R of

$$\frac{1}{N}\sum_{i=1}^{N} ((Bf)(t_i)-y(t_i))^2 + \lambda||f||_R^2 .$$

($f_{N,\lambda}$ is a regularized solution to (6.1).). Let $g_{N,\lambda}$ be the minimizer in H_K of

$$\frac{1}{N}\sum_{i=1}^{N} (g(t_i)-y(t_i))^2 + \lambda||g||_K^2 .$$

It is not hard to show that $B^+g_{N,\lambda} \equiv f_{N,\lambda}$. Thus, one might wish to consider designs which minimize

$$E||g-g_{N,\lambda}||_R^2 \equiv E||B^+Bf-f_{N,\lambda}||_R^2 .$$

Appendix

Proof of Lemma 3.3.

We only prove (3.15), the proof of (3.14) is similar.

First, it can be verified that if P is an orthogonal projector in an r.k.h.s. H_R, and $\rho \in L_2(T)$, and

$$f(\tau) = \int R(\tau,\sigma)\rho(\sigma)d\sigma ,$$

that is

$$f(\cdot) = \int R_\sigma(\cdot)\rho(\sigma)d\sigma ,$$

then

$$(Pf)(\cdot) = \int PR_\sigma(\cdot)\rho(\sigma)d\sigma$$

and

$$||Pf||_R^2 = \iint \langle R_\sigma, PR_\tau\rangle_R \rho(\sigma)\rho(\tau)d\sigma d\tau .$$

Letting $\underset{\sim}{t} = (x,y)$, $\underset{\sim}{s} = (u,v)$ gives

$$\langle K_t, R_x^{N_1} R_y^{N_2} K_t\rangle_K = \langle (I-A_{N_1})Q_x, Q_u\rangle_Q \langle (I-A_{N_2})Q_y, Q_v\rangle_Q$$

and

$$||R_x^{N_1} R_y^{N_2} f||_K^2 = \int_{x,y,u,v}\!\!\int\!\!\int\!\!\int \rho(x,y)\rho(u,v)[Q(x,u)-\hat{Q}_1(x,u)] \cdot$$

$$[Q(y,v)-\hat{Q}_2(y,v)]dxdydudv , \qquad \text{(A.1)}$$

where we are denoting $\langle A_{N_1}Q_x, Q_u\rangle_Q$ by $\hat{Q}_1(x,u)$ and $\langle A_{N_2}Q_y, Q_v\rangle_Q$ by $\hat{Q}_2(y,v)$. Without loss of generality let $N_1 = 2mL_1$, $N_2 = 2mL_2$, where L_1 and L_2 are integers. Following [23], the x,u square is divided into L_1^2 squares $I_{ij} = [\frac{i-1}{L_1}, \frac{i}{L_1}]\times[\frac{j-1}{L_1}, \frac{j}{L_1}]$, $i, j = 1,2,\ldots,L_1$, similarly, the y,v square is divided into L_2^2 squares, $I_{k\ell} = [\frac{k-1}{L_2}, \frac{k}{L_2}]\times[\frac{\ell-1}{L_2}, \frac{\ell}{L_2}]$, $k,\ell = 1,2,\ldots,L_2$. It is shown in [23], that, if Q has property m, then there exists constants δ_1 and δ_2 such

that

$$\sup_{x,u\in I_{ij}} |Q(x,u)-\hat{Q}_1(x,u)| \le \frac{\delta_1}{N_1^{2m}}, \quad i \neq j \tag{A.2}$$

$$\le \frac{\delta_2}{N_1^{2m-1}}, \quad i = j \tag{A.3}$$

and similarly for $\hat{Q}_2(y,v)$.

Letting

$$\nu_{ik} = \int_{x_i}^{x_{i+1}} \int_{y_k}^{y_{k+1}} |\rho(x,y)|dxdy ,$$

where $x_i = \frac{i-1}{N_1}$, $y_k = \frac{k-1}{N_2}$, gives, from (A.1-A.3)

$$||R_x^{N_1} R_y^{N_2} f||_K^2 \le \frac{\delta_1^2}{N_1^{2m}N_2^{2m}} \sum_{i\neq j} \sum_{k\neq \ell} \nu_{ik}\nu_{j\ell}$$

$$+ \frac{\delta_1\delta_2}{N_1^{2m}N_2^{2m-1}} \sum_{i\neq j} \sum_{k} \nu_{ik}\nu_{jk}$$

$$+ \frac{\delta_1\delta_2}{N_1^{2m-1}N_2^{2m}} \sum_{i} \sum_{k\neq \ell} \nu_{ik}\nu_{i\ell}$$

$$+ \frac{\delta_2^2}{N_1^{2m-1}N_2^{2m-1}} \sum_{i} \sum_{k} \nu_{ik}^2 \tag{A.4}$$

Now

$$\nu_{ik}\nu_{j\ell} \le \frac{1}{L_1L_2} \mu_{ik}^{1/2} \mu_{j\ell}^{1/2}$$

where

$$\mu_{ik} = \int_{x_i}^{x_{i+1}} \int_{y_k}^{y_{k+1}} \rho^2(x,y)dxdy.$$

giving

$$\sum_{ij} \sum_{k\ell} \nu_{ik}\, \nu_{j\ell} = [\int_0^1\int_0^1 |\rho(x,y)|dxdy]^2 \le \int_0^1\int_0^1 \rho^2(x,y)dxdy$$

$$L_1L_2 \sum_{i,k} \nu_{ik}^2 \le \sum_{ik} \mu_{ik} = \int_0^1\int_0^1 \rho^2(x,y)dxdy$$

$$L_2 \sum_{ijk} \nu_{ik}\, \nu_{jk} \le \frac{1}{L_1} \sum_{ijk} \mu_{ik}^{1/2} \mu_{jk}^{1/2}$$

$$\le \frac{1}{L_1} \sum_{ij} \sqrt{\sum_k \mu_{ik}} \sqrt{\sum_k \mu_{jk}}$$

$$= \frac{1}{L_1} (\sum_i \sqrt{\sum_k \mu_{ik}})^2 \le \sum_i \sum_k \mu_{ik}$$

and similarly for the third term on the right of (A.4). Therefore

$$||R_x^{N_1} R_y^{N_2} f||_K^2 \le \frac{\delta_1^2+2m\delta_1\delta_2+(2m)^2\delta_2^2}{N_1^{2m}N_2^{2m}} \int_0^1\int_0^1 \rho^2(u,v)dudv .$$

Letting $e_2 = \delta_1^2 + 2m\delta_1\delta_2 + (2m)^2\delta_2^2$ gives (3.15).

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We consider the problem of interpolating a function on the unit square from values of the function given on a set (called the "design") of N discrete points. We obtain some new results on better ways to distribute these N points so as to minimize the maximum interpolation error. W. J. Gordon [Approximation With Special Emphasis on Spline Functions, I. Schoenberg, ed. 1969], showed that an improvement on the tensor product design $\{(x,y)\} = \{\frac{i}{n}, \frac{j}{n}, i,j = 1,2,\ldots,n\}$, of $N = n^2$ interpolation points could be made by using a design that is the union of two tensor product designs: $\{(x,y)\} = \{\frac{i}{n}, \frac{j}{n^2}, i = 1,2,\ldots,n, j = 1,2,\ldots,n^2\} \cup \{\frac{i}{n^2}, \frac{j}{n}, i = 1,2,\ldots,n^2, j = 1,2,\ldots,n\}$. Here $N = 2n^3 - n^2$ since the intersection of these tensor product designs has n^2 points. Gordon showed that where interpolation on the original tensor product design has a convergence rate $O(N^{-p/2})$ for some p with respect to functions with specified smoothness properties, the new design will have an associated convergence rate of $O(N^{-2p/3})$. We construct the ℓth design which is the union of $\ell+1$ tensor product designs: $\{(x,y)\} = \bigcup_{k=1}^{\ell+1} \frac{i}{n^k}, \frac{j}{n^{\ell+2-k}}, i = 1,2,\ldots,n^k, j = 1,2,\ldots,n^{\ell+2-k}$. Here the ℓth design has $N = (\ell+1)n^{\ell+2} - \ell n^{\ell+1}$ distinct points, and we show that the associated convergence rate is $O(N^{-[(\ell+1)p/(\ell+2)]})$. Some exact theorems to this effect are proved for convergence of minimal norm interpolation in tensor product reproducing kernel Hilbert spaces. An application to the reduction of X-ray dosage in the reconstruction of human internal structure from X-ray projections is suggested. Applications to the design of certain (other) indirect sensing experiments are also noted.

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